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A PHYSICAL APPROACH TO ASYMPTOTICALLY SOLVING SOME INTEGRAL EQUATIONS OCCURRING IN SOLID MECHANICS

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Abstract—We use the Comninou model of the interface crack to illustrate a method of solving a class of integral equations containing a small parameter. We need to know the inverse of the integral equation when this small parameter is set equal to zero. This inverse, when applied to the original integral equation, leads to an expression for the unknown which is an asymptotic series in the small parameter, provided the field point x is outside the boundary layer. Next we need to know the inverse of an integral equation which models the given integral equation for points x in the boundary layer. When this inverse is applied to the original integral equation, one obtains an expression for the unknown which is an asymptotic series in the small parameter, provided the field point x is inside the boundary layer. This solution and the former solution are often referred to as the inner and outer solutions, respectively. They each contain unknown constants. These constants are determined by matching the inner and outer solutions on an overlap region where both solutions are valid. A uniform expression for the solution is obtained by adding the inner and outer solutions and subtracting off the matching terms.

1. INTRODUCTION

We describe a physical approach to finding the asymptotic solutions to some integral equations occurring in solid mechanics. These equations should have a small parameter and a boundary layer. We need to know the inverse of the problem when the small parameter is zero. Usually, this problem has some physical significance. We then extend the given integral equation to this case by adding an unknown forcing function. When this problem is inverted, the inverse operating on the unknown forcing function turns out to be separable away from the boundary layer. This leads to an outer solution of the form $\psi^o(x)+\Sigma_{n=1}^{\infty} f_n^o\psi_n^o(x)$, where $\psi^o(x)$ and $\psi_n^o(x)$ are known functions and f_n^o are unknown constants. These constants are functionals of the unknown forcing function. The series for the outer solution is asymptotic with respect to the small parameter.

Next we need to find the solution valid in a neighborhood of the boundary layer. We assume that we can extend the integral by adding an unknown forcing function in such a way that the inverse is known and the resulting solution is valid in the boundary layer. Usually, this can be done in a manner similar to that for the outer solution. Again the inverse operating on the unknown forcing function should be separable in the boundary layer. This leads to an inner solution of the form $\psi^i(x)+\Sigma_{n=0}^{\infty} f_n^i\psi_n^i(x)$, where $\psi^i(x)$ and $\psi_n^i(x)$ are known functions and f_n^i are unknown constants. Again, the series is asymptotic with respect to the small parameter.

The unknown constants are determined by matching the inner and outer solutions on an overlap region. A uniform solution is then obtained by adding the inner and outer solutions and subtracting off the matching terms.

We illustrate this approach by solving the integral equations corresponding to the Comninou (1977, 1978) model of the interface crack. In this problem the small parameter is the extent of the left contact zone. The integral equation for the outer solution is obtained by setting the extent of this contact zone equal to zero. The equality of the integral equation is preserved by adding the unknown normal traction in this contact zone to the forcing

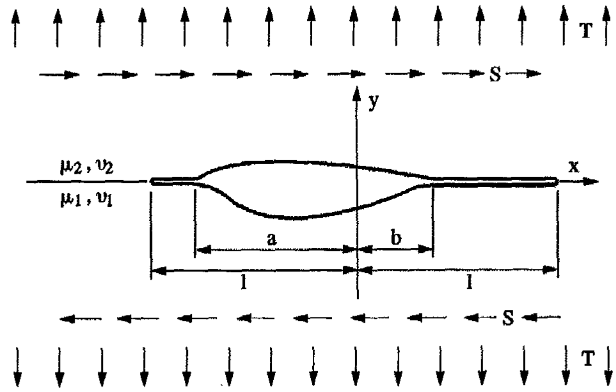

Fig. 1. The interface crack.

function. This integral equation has been solved by Dundurs and Gautesen (1988). Physically, we expect that away from the boundary layer the effect of the normal traction in the left contact zone on the outer solution is small.

To obtain an integral equation governing the inner solution we set the extent of the right contact zone equal to zero and preserve the equality by adding the unknown normal traction in this contact zone to the forcing function. As this integral equation is analogous to the integral equation governing the outer solution, its inverse is known. In this case, we expect the normal traction in the right contact zone to have an $O(1)$ effect on the inner solution in the boundary. However, except for its magnitude, we expect that the functional form of this effect is known.

In this problem, the series which appear in the inner and outer solutions converge uniformly. Also, in this problem we must determine the extents of the left and right contact zones. This is done after the matching procedure by imposing an auxiliary condition and ensuring that a constant determined by matching is consistent with its definition.

We mention some related work on the interface crack. Gautesen and Dundurs (1987, 1988) have given exact solutions to the integral equations. However, these solutions are extremely complex and difficult to work with. Subsequently, Gautesen (1992, 1993) has shown that the gap is the solution to an eigenvalue problem. Also, he has given simple leading order asymptotic expressions for most quantities of physical interest as well as representative graphs. We do not repeat these results here. However, our results are, of course, in agreement with his.

In the next section we obtain the integral equation governing this problem. In section 3, we give the integral equations governing the inner and outer solutions as well as their solutions. Then to leading order, we determine the unknown constants by matching the inner and outer solutions. Finally, we determine the extents of the left and right contact zones. In the last section, we discuss our results.

2. THE INTEGRAL EQUATION

We use the Comninou (1977, 1978) model of the interface crack. In this model the tips of the cracks are assumed closed (see Fig. 1). If the crack is allowed to be fully open, then one experiences the contradiction that there is interpenetration of material near the crack tips (England, 1965; Malyshev and Salganik, 1965). The governing integral equations for this problem are

$$\beta B_x(x) + \frac{1}{\pi}\int_{-a}^{b} \frac{B_y(\xi)\mathrm{d}\xi}{\xi - x} = \frac{T}{C}, \quad -a < x < b \tag{1}$$

$$-\beta B_y(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{B_x(\xi)\mathrm{d}\xi}{\xi - x} = \frac{S}{C}, \quad -1 < x < 1, \tag{2}$$

where $B_x(x)$ and $B_y(x)$ are dislocation densities of the glide and climb arrays, and the climb array is, of course, identically zero in the contact zones:

$$B_y(x) = 0, \quad -1 < x < -a \quad \text{or} \quad b < x < 1. \tag{3}$$

Also, T and S denote the applied tension and shear at infinity, and C and the Dundurs parameters, α and β, are related to the shear modulus μ by

$$C = \frac{2\mu_1(1+\alpha)}{(1-\beta^2)(\kappa_1+1)} = \frac{2\mu_2(1-\alpha)}{(1-\beta^2)(\kappa_2+1)} \tag{4}$$

$$\alpha = \frac{\mu_2(\kappa_1+1)-\mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)} \tag{5}$$

$$\beta = \frac{\mu_2(\kappa_1-1)-\mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)}, \tag{6}$$

where $\kappa = 3-\nu$ for plane strain, with ν denoting Poisson's ratio.

The dislocation densities must also satisfy

$$\int_{-1}^{1} B_x(x)\,\mathrm{d}x = 0, \quad \int_{-a}^{b} B_y(x)\mathrm{d}x = 0 \tag{7}$$

to ensure that the tangential shift vanishes at $x = \pm 1$ and that the gap vanishes at $x = b, -a$. In addition, the gap must be positive on the open portion of the crack, $-a < x < b$, while the normal traction must be compressive in the contact zones, $-1 < x < -a$ or $b < x < 1$. Dundurs and Comninou (1979) have noted that these inequalities imply that the climb array must also satisfy

$$B_y(-a) = B_y(b) = 0. \tag{8}$$

For convenience, we have taken the length of the crack to be 2. Without loss of generality, we need only consider the case $S > 0$ and $\beta > 0$ (Comninou and Schmueser, 1979). The mismatch in elastic constants has been assumed to be such that the extent of the left contact zone is smaller than the extent of the right contact zone.

We solve eqn (2) for $B_x(x)$, imposing the auxiliary condition (7). The result is

$$B_x(x) = \frac{Sx}{C(1-x^2)^{1/2}} - \frac{\beta}{\pi}\int_{-a}^{b}\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\frac{B_y(\xi)\mathrm{d}\xi}{\xi-x}. \tag{9}$$

Then, upon substitution into eqn (1), we obtain the integral equation governing the climb array:

$$\frac{1}{\pi}\int_{-a}^{b}\left[1-\beta^2\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\right]\frac{B_y(\xi)\mathrm{d}\xi}{\xi-x} = F(x), \quad -a < x < b, \tag{10}$$

where

$$F(x) = \frac{T}{C} - \frac{\beta S x}{C(1-x^2)^{1/2}}. \tag{11}$$

3. ASYMPTOTIC SOLUTION TO THE INTEGRAL EQUATION

The integral equation (10) has been solved in closed form by Gautesen and Dundurs (1988). However their solution is very complex and difficult to work with. We wish to take advantage of the fact that the extent δ of the left contact zone is very small:

$$\delta = 1-a \ll 1. \tag{12}$$

Dundurs and Gautesen (1988) have solved eqn (10) with $a = 1$. Their results for the climb array $B_y(x)$ were good provided that $1+x \gg \delta$, and they obtained an excellent approximation to the extent of the right contact zone.

Thus it seems natural to extend the integral equation (10) to the larger interval $-1 < x < b$ in such a way that we preserve the equality. Thus we write

$$\frac{1}{\pi}\int_{-1}^{b}\left[1-\beta^2\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\right]\frac{B_y(\xi)\mathrm{d}\xi}{\xi-x} = F_o(x), \quad -1 < x < b, \tag{13}$$

where

$$F_o(x) = F(x) - C^{-1}\sigma_y^-(x) \tag{14}$$

$$\sigma_y^-(x) = 0, \quad -a < x < b. \tag{15}$$

Since $B_y(x)$ vanishes in the left contact zone, eqn (13) agrees with eqn (10) on the open portion of the crack, $-a < x < b$. We choose $\sigma_y^-(x)$ so that the equality in eqn (13) is preserved for $-1 < x < -a$. Physically, $\sigma_y^-(x)$ represents the normal traction in the left contact zone. It should have little effect on the climb array except when x is close to $-a$.

From Dundurs and Gautesen (1988), we find that the solution to eqn (13), which satisfies $B_y(b) = 0$, is

$$\pi(1-\beta^2)B_y(x) = \int_{-1}^{b}\frac{(1-\xi^2)^{1/2}F_o(\xi)}{t_o(1-x^2)^{1/2}(\xi-x)}(z_o\phi_r(z_o)\phi_r(t_o)+t_o\phi_i(z_o)\phi_i(t_o))\mathrm{d}\xi \tag{16}$$

$$z_o = \left|\frac{2(b-x)}{(1+b)(1-x)}\right|^{1/2}, \quad t_o = \left[\frac{2(b-\xi)}{(1+b)(1-\xi)}\right]^{1/2} \tag{17}$$

$$\phi_r(z)+i\phi_i(z) = \phi(z) = \left|\frac{1+z}{1-z}\right|^{i\beta_1}, \quad \beta_1 = \frac{1}{2\pi}\log\left(\frac{1+\beta}{1-\beta}\right); \tag{18}$$

$i = (-1)^{1/2}$ and $\phi_r(z)$ and $\phi_i(z)$ are real-valued functions of z. When $F_o = F$, the integral in eqn (16) can be evaluated in closed form. When $F_o = -\sigma_y^-/C$ in eqn (16), the integral is small except when x is close to $-a$. To evaluate this integral, we require that $1+x \gg \delta$ and substitute the identity

$$\frac{1}{\xi-x} = -\frac{1}{1+x}\sum_{n=0}^{\infty}\left(\frac{1+\xi}{1+x}\right)^n, \quad 0 < 1+\xi < \delta, \quad \delta \ll 1+x. \tag{19}$$

The result is

$$(1-\beta^2)^{1/2}(1-x^2)^{1/2}CB_y(x) = z_o\phi_r(z_o)\left[\beta_1(1+b)g_i+\gamma_b^{-1}(x+\tfrac{1}{2}b-\tfrac{1}{2})g_r+\sum_{n=1}^{\infty}\delta^n G_n^r(1+x)^{-n}\right]$$

$$+\phi_i(z_o)\left[xg_i-2\beta_1\gamma_b^{-1}g_r+\sum_{n=1}^{\infty}\delta^n G_n^i(1+x)^{-n}\right], \quad \delta \ll 1+x<1+b, \quad (20)$$

where g_r and g_i are real constants defined by

$$g_r+ig_i = (T+iS)\phi(\gamma_b), \quad (21)$$

and G_n^r and G_n^i are real-valued constants defined by

$$\pi(1-\beta^2)^{1/2}(G_n^r+iG_n^i) = -\delta^{-n}\int_{-1}^{-a}(1-\xi^2)^{1/2}(1+\xi)^{n-1}[t_o^{-1}\phi_r(t_o)+i\phi_i(t_o)]\sigma_y^-(\xi)\mathrm{d}\xi \quad (22)$$

$$\gamma_b = [2/(1+b)]^{1/2}. \quad (23)$$

We remark that, with respect to δ, G_n^r and G_n^i are $O(1)$, since $(1+\xi)^{1/2}\sigma_y^-(\xi) = O(1)$ in the left contact zone.

Thus far, we have an expression (20) for $B_y(x)$ valid for x far from the left contact zone. The procedure for finding the unknown constants G_n will be described below. We now seek an expression for $B_y(x)$ valid for x near the left contact zone. To this end, we extend the integral equation (10) to the region $-a < x < 1$ by

$$\frac{1}{\pi}\int_{-a}^{1}\left[1-\beta^2\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\right]\frac{B_y(\xi)\mathrm{d}\xi}{\xi-x} = F_i(x), \quad -a<x<1, \quad (24)$$

where

$$F_i(x) = F(x)-C^{-1}\sigma_y^+(x) \quad (25)$$

$$\sigma_y^+(x) = 0, \quad -a<x<b \quad (26)$$

and $F(x)$ is given by eqn (11). Since $B_y(x)$ vanishes in the right contact zone, eqn (24) agrees with eqn (10) on the open portion of the crack, $-a<x<b$. We choose $\sigma_y^+(x)$ such that the equality in eqn (24) is preserved for $b<x<1$. Again we can physically interpret $\sigma_y^+(x)$ as the normal traction in the right contact zone.

The inverse of eqn (24), satisfying $B_y(-a) = 0$, is

$$\pi(1-\beta^2)B_y(x) = \int_{-a}^{1}\frac{(1-\xi^2)^{1/2}F_i(\xi)}{t_i(1-x^2)^{1/2}(\xi-x)}(z_i\phi_r(z_i)\phi_r(t_i)+t_i\phi_i(z_i)\phi_i(t_i))\mathrm{d}\xi, \quad (27)$$

where

$$z_i = \left[\frac{2(a+x)}{(1+a)(1+x)}\right]^{1/2}, \quad t_i = \left[\frac{2(a+\xi)}{(1+a)(1+\xi)}\right]^{1/2} \quad (28)$$

and ϕ_r and ϕ_i are given by eqn (18). When $F_i = F$ in eqn (27), the integral can be evaluated in closed form. When $F_i = -\sigma_y^+/C$, we evaluate the integral by requiring that $1+x \ll 1$ and substituting the identity

$$\frac{1}{\xi-x}=\frac{1}{1+\xi}\sum_{n=0}^{\infty}\left(\frac{1+x}{1+\xi}\right)^n,\quad \delta<1+x\ll 1. \tag{29}$$

The result is

$$(1-\beta^2)^{1/2}(1-x^2)^{1/2}CB_y(x)=z_i\phi_r(z_i)\left[H_r-(1+x)\gamma_a^{-1}h_r+\sum_{n=1}^{\infty}(1+x)^nH_n^r\right]$$
$$+\phi_i(z_i)\left[H_i-(1+x)h_i+\sum_{n=1}^{\infty}(1+x)^nH_n^i\right],\quad \delta<1+x\ll 1, \tag{30}$$

where h_r and h_i are real-valued constants defined by

$$h_r+ih_i=(-T+iS)\phi(\gamma_a) \tag{31}$$

$$\gamma_a=[2/(1+a)]^{1/2}, \tag{32}$$

and H_n^r, H_n^i, H_r and H_i are real-valued constants defined by

$$\pi(1-\beta^2)^{1/2}(H_n^r+iH_n^i)=\int_b^1(1-\xi^2)^{1/2}(1+\xi)^{-n-1}[t_i^{-1}\phi_r(t_i)+i\phi_i(t_i)]\sigma_y^+(\xi)\mathrm{d}\xi \tag{33}$$

$$H_r+iH_i=H_0^r+iH_0^i+(1-2i\beta_1)(\gamma_a^{-1}h_r+ih_i)-\tfrac{1}{2}\delta(\gamma_a^{-1}h_r-2i\beta_1h_i). \tag{34}$$

We remark that the constants defined by eqns (33) and (34) are $O(1)$ with respect to δ.

It remains to determine the unknown constants and to determine the extents of the left and right contact zones. We illustrate the procedure by considering only the leading order terms in the expressions (20) and (30) for $B_y(x)$. Thus we take

$$(1-\beta^2)^{1/2}(1-x^2)^{1/2}CB_y(x)\sim z_o\phi_r(z_o)\,[\beta_1(1+b)g_i+\gamma_b^{-1}(x-\tfrac{1}{2}+\tfrac{1}{2}b)g_r]$$
$$+\phi_i(z_o)[xg_i-2\beta_1\gamma_b^{-1}g_r],\quad \delta\ll 1+x<1+b \tag{35}$$

$$(1-\beta^2)^{1/2}(1-x^2)^{1/2}CB_y(x)\sim z_i\phi_r(z_i)H_r+\phi_i(z_i)H_i,\quad \delta<1+x\ll 1. \tag{36}$$

Both solutions are valid on the overlap region

$$1+x=\delta^{1/2}y,\quad y=O(1). \tag{37}$$

Matching eqns (35) and (36) on this overlap region yields

$$H_r-iH_i=\exp\left[-i\beta_1\log\left(\frac{32(1+b)}{\delta(1-b)}\right)\right]\left\{\gamma_b^{-1}g_r\left(\frac{1}{2}b-\frac{3}{2}-2i\beta_1\right)+g_i(\beta_1+b\beta_1-i)\right\}. \tag{38}$$

It remains to determine the extent of the left and right contact zones. First, we observe that the constants H_r and H_i determined by matching [see eqn (38)] must be consistent with their definitions in eqns (34) and (33). To leading order we find that

$$g_r\sim-2\beta_1\gamma_bg_i. \tag{39}$$

From eqn (21) we find that b satisfies

$$2\beta_1 \operatorname{arctanh}(\gamma_b^{-1}) - \arctan(2\beta_1\gamma_b) - \arctan(T/S) = 0 \tag{40}$$

and

$$g_i = [(T^2+S^2)/(1+4\beta_1^2\gamma_b^2)]^{1/2}. \tag{41}$$

There are many values of b for which eqn (39) is satisfied. The value of b obtained from eqn (40) is the only one for which the normal traction in the right contact zone is compressive:

$$(1-x^2)^{1/2}\sigma_y^+(x) \sim -(1-\beta^2)^{1/2}g_i[(x+4\beta_1^2)\sinh(\arctan(z_o))+2\beta_1 z_o\cosh(\arctan(z_o))]. \tag{42}$$

Unfortunately, we must numerically solve eqn (40) for γ_b. Then b follows from eqn (23).

To determine the extent of the right contact zone, we impose the auxiliary condition (7) on $B_y(x)$. This leads to

$$H_r = 0, \quad H_i = -g_i(1+4\beta_1^2) \tag{43}$$

$$(1-a) = 32\left(\frac{1+b}{1-b}\right)\exp[-\beta_1^{-1}(\pi+2\arctan(2\beta_1))]. \tag{44}$$

Finally, we obtain a uniform expression for $B_y(x)$. The matching term can be expressed as

$$(1-\beta^2)^{1/2}(1-x^2)^{1/2}CB_y(x) \sim -(1+4\beta_1^2)g_i\sin(\beta_1\log[4\delta^{-1}(1+x)]), \quad 1+x = O(\delta^{1/2}). \tag{45}$$

Subtracting eqn (45) from the sum of eqns (35) and (36) yields an expression for $B_y(x)$ which is uniformly valid for $-a < x < b$.

4. DISCUSSION

The general procedure for determining $B_y(x)$ is as follows. The sums in the expressions for the outer solution [eqn (20)] and the inner solution [eqn (30)] are terminated after N terms. The constants G_n^r, G_n^i, $n = 1, 2, \ldots, N$ and H_n^r, H_n^i, $n = 0, 1, \ldots, N$ are determined by matching the inner and outer solutions on the region where $(1+x) = O(\delta^{1/2})$. The matching terms have the form

$$\operatorname{Re}\left(\exp[i\beta_1\log(1+x)]\left\{c_0+\sum_{n=1}^{N}c_n(1+x)^n+d_n\delta^n(1+x)^{-n}\right\}\right), \tag{46}$$

where c_n and d_n are complex-valued constants.

One relation between a and b is obtained by making the constants H_r and H_i, determined by matching, consistent with their definition [eqn (34)]. Next one imposes the condition $\int_{-a}^{b}B_y(x)\,dx = 0$. This yields the other relation between the constants a and b. A uniform expression for $B_y(x)$ is obtained by adding the inner and outer solutions and subtracting the matching terms.

We now discuss the convergence of the series in eqns (20) and (30). From eqns (22) and (33) we easily obtain the following estimates:

$$\pi(1-\beta^2)^{1/2}\,|G_n^r+iG_n^i| \leqslant \hat{t}n^{-1} \max_{-1<\xi<-a} |(1-\xi^2)^{1/2}\sigma_y^-(\xi)| \tag{47}$$

$$\pi(1-\beta^2)^{1/2}\,|H_n^r+iH_n^i| \leqslant \hat{t}n^{-1}(1+b)^{-n} \max_{b<\xi<1} |(1-\xi^2)^{1/2}\sigma_y^+(\xi)|, \quad n \geqslant 1 \tag{48}$$

$$\hat{t} = \left[\frac{(1+a)(1+b)}{2(a+b)}\right]^{1/2}. \tag{49}$$

These estimates imply that the series in (20) converges for $-a < x \leqslant b$ and that the series in eqn (30) converges for $-a \leqslant x < b$.

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REFERENCES

Comninou, M. (1977). The interface crack. *ASME J. Appl. Mech.* **44,** 631–636.

Comninou, M. (1978). The interface crack in a shear field. *ASME J. Appl. Mech.* **45,** 287–290.

Comninou, M. and Schmueser, D. (1979). The interface crack in a combined tension–shear field. *ASME J. Appl. Mech.* **46,** 345–348.

Dundurs, J. and Comninou, M. (1979). Some consequences of the inequality conditions in contact and crack problems. *J. Elasticity* **9,** 71–82.

Dundurs, J. and Gautesen, A. K. (1988). An opportunistic analysis of the interface crack. *Int. J. Fract.* **36,** 151–159.

England, A. H. (1965). A crack between dissimilar media. *ASME J. Appl. Mech.* **32,** 400–402.

Gautesen, A. K. (1992). The interface crack in a tension field: an eigenvalue problem for the gap. *Int. J. Fract.* **55,** 261–271.

Gautesen, A. K. (1993). The interface crack under combined loading: an eigenvalue problem for the gap. *Int. J. Fract.* **60,** 349–361.

Gautesen, A. K. and Dundurs, J. (1987). The interface crack in a tension field. *ASME J. Appl. Mech.* **54,** 93–98.

Gautesen, A. K. and Dundurs, J. (1988). The interface crack under combined loading. *ASME J. Appl. Mech.* **55,** 580–586.

Malyshev, B. M. and Salganik, R. L. (1965). The strength of adhesive joints using the theory of fracture. *Int. J. Fract. Mech.* **1,** 114–128.